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The stability of non-dissipative Couette flow with an axial magnetic field

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The criterion for stability has been derived by an examination of the stability of the basic flow to arbitrary perturbations. It has been proved by variational method that the system is always unstable under non-symmetric perturbations The characteristic equation has been derived for asymmetric perturbations and the minimum field strength required to stabilize the system against axisymmetric perturbations has been obtained by using the theory of finite Hankel transforms The perturbation equations have been solved under the plane layer approximations, when the cylinders are counter rotating or rotating with greatly differing angular velocities

1. INTRODUCTION

The stability of non-dissipative Couette flow with an axial magnetic field, in which the boundaries are two coaxial cylinders and the basic flow is circular and the effects of viscosity and resistivity are negligible, was considered by Chandrasekhar. (1960a). He established the criterion for stability by examining the stability of basic flow for arbitrary perturbations. Analysing the disturbance into normal modes he obtained the solutions of the perturbation equations, whose dependence on t, θ and z is given by, $\exp[i(pt+m\theta+kz)]$, where p is a constant (which can be complex), m is an integer (positive, zero or negative), and k is the wave number of the disturbance in the z-direction. The nonmagnetic case ($m \neq 0$) has been examined by Chandrasekhar (1960b). In this paper the stability has been examined for general values of m by using a variational method of Chandrasekhar (1960b) and also by solving the perturbation equations by means of finite Hankel transforms (Sneddon 1961). The precise stability criterion has also been derived for axisymmetric perturbations. Further the perturbation equations have been solved under the narrow gap approximations when the cylinders are counter-rotating or rotating with greatly differing angular velocities.

2. Equations of the Problem

The perturbation equations are (Chandrasekhar 1961, p 395 equations (4.4), (4.7))

$$(\sigma^2-\Omega_A^2)D_*\xi_r-2m\sigma\frac{\Omega}{r}\xi_r=\left(\frac{m^2}{r^2}+k^2\right)\tilde{\omega}\,. \quad \dots \quad (1)$$

$$\left(\sigma^2-\Omega_A^2-\phi(r)-\frac{4\Omega^2\Omega_A^2}{\sigma^2-\Omega_A^2}\right)\xi_r=D\tilde{\omega}+\frac{2m\sigma\Omega}{\sigma^2-\Omega_A^2}\frac{\tilde{w}}{r}\,, \quad \dots \quad (2)$$

with the boundary conditions

$$\xi_r=0 \quad \text{for} \quad r=R_1 \quad \text{and} \quad R_2. \quad \dots \quad (3)$$

A useful relation in this connection is (Chandrasekhar 1961, p 385 equation (4.0))

$$(\sigma^2-\Omega_A^2)\xi_z=ik\tilde{\omega}. \quad \dots \quad (4)$$

In the above equations all the notations used have their usual meaning as described in Chandrasekhar's above (paper pp. 384-5)

Chandrasekhar has stated that equations (1) and (2) represent a self-adjoint system with respect to the boundary conditions (3) and that in consequence its solution can be reduced to a variational problem. Indeed, by the same procedure as followed by Chandrasekhar (1960b) it is easy to verify that for fixed p and m, this can be regarded as a self-adjoint eigenvalue problem for k^2 and k^2 can be determined by variational methods as the ratio of two integrals.

Thus,

$$-k^2=\frac{\int_{R_1}^{R^2}\left[\frac{f(\sigma)}{\sigma^2-\Omega_A^2}\xi_r^2+\frac{m^2\tilde{\omega}^2}{\sigma^2-\Omega_A^2}\frac{1}{r^2}\right]r\mathrm{d}r}{\int_{R_1}^{R^2}\frac{r\tilde{\omega}^2}{\sigma^2-\Omega_A^2}\,\mathrm{d}r}, \quad \dots \quad (5)$$

where

$$f(\sigma)=\sigma^4-\sigma^2(\phi+2\Omega_A^2)+\Omega_A^2(\phi+\Omega_A^2-4\Omega^2). \quad \dots \quad (6)$$

Further, putting the value of $\tilde{\omega}$ from (4) in (1), (2) and (5), we obtain

$$ik\left[(\sigma^2-\Omega_A^2)D_*\xi_r-2m\sigma\frac{\Omega}{r}\xi_r\right]=\left(\frac{m^2}{r^2}+k^2\right)(\sigma^2-\Omega_A^2)\xi_z\,, \quad \dots \quad (7)$$

$$\frac{ikf(\sigma)}{\sigma^2-\Omega_A^2}\xi_r=D[(\sigma^2-\Omega_A^3)\xi_z]+2m\sigma\frac{\Omega}{r}\xi_z, \quad \dots \quad (8)$$

and
$$-\frac{m^2}{k^2}=\frac{\int_{R_1}^{R_2}\left[\frac{-f(\sigma)}{\sigma^2-\Omega_A^2}\xi_r^2+(\sigma^2-\Omega_A^2)\xi_z^2\right]r\mathrm{d}r}{\int_{R_1}^{R_2}\frac{\sigma^2-\Omega_A^2}{r}\xi_z^2\mathrm{d}r}=\frac{I_1}{I_2}\ \text{(say)}. \quad \dots \quad (9)$$

Now we shall show that this equation also provides the basis for a variational determination of k^2 for assigned values of the other parameters.

First, it should be remarked that in evaluating k^2 in accordance with (9), it is to be assumed that ξ_r is determined in terms of ξ_z by equation (8) and that an "allowed" ξ_z accordingly satisfies the boundary conditions

$$\mathrm{D}[(\sigma^2-\Omega_A{}^2)\xi_z]+\frac{2m\sigma\Omega}{r}\,\xi_z=0 \quad \text{for } r=R_1 \text{ and } R_2. \qquad \dots \quad (10)$$

Now consider the effect on k^2 of the arbitrary variation $\delta\xi_z$ in ξ_z compatible with the boundary conditions (10). If $\delta\xi_r$ denotes the corresponding variation in ξ_r, then by (8)

$$\frac{if(\sigma)}{\sigma^2-\Omega_A{}^2}\left[\frac{1}{2k}\delta k^2.\xi_r+k\delta\xi_r\right]=\mathrm{D}\{(\sigma^2-\Omega_A{}^2)\delta\xi_z\}+\frac{2m\sigma\Omega}{r}\,\delta\xi_z, \qquad (11)$$

and $\delta\xi_r=0$ for $r=R_1$ and R_2.

From equation (9) it follows that

$$ \qquad (12)$$

where

$$\delta I_1=2\int_{R_1}^{R_2}\left[\frac{-f(\sigma)}{\sigma^2-\Omega_A{}^2}\,\xi_r\delta\xi_r+(\sigma^2-\Omega_A{}^2)\xi_z\delta\xi_z\right]r\mathrm{d}r \qquad (13)$$

and
$$\delta I_2=2\int_{R_1}^{R_2}\frac{\sigma^2-\Omega_A{}^2}{r}\,\xi_z\delta\xi_z\mathrm{d}r \qquad (14)$$

are the variations in I_1 and I_2 resulting from the variations in ξ_z.

The terms in $\delta\xi_r$ in the expression (13) for δI_1 are the same as

$$2\int_{R_1}^{R_2} r\xi_r\left[\frac{i}{k}\mathrm{D}[\sigma^2-\Omega_A{}^2)\delta\xi_z]-\frac{2m\sigma\Omega}{rik}\delta\xi_z+\frac{f(\sigma)}{\sigma^2-\Omega_A{}^2}\cdot\frac{\delta k^2}{2k^2}\,\xi_r\right]\mathrm{d}r. \qquad \dots \quad (15)$$

After an integration by parts (15) can be rewritten as

$$-2\int_{R_1}^{R_2}\left[(\sigma^2-\Omega_A{}^2)\frac{i}{k}\delta\xi_z\frac{\mathrm{d}}{\mathrm{d}r}(r\xi_r)+\xi_r\frac{2m\sigma\Omega}{ik}\delta\xi_z-\frac{f(\sigma)}{\sigma^2-\Omega_A{}^2}\,\frac{\delta k^2}{2k^2}\,r\xi_r{}^2\right]\mathrm{d}r. \quad \dots \quad (16)$$

The first order change in k^2 resulting from the variation in ξ_z is, therefore,

$$\begin{aligned} m^2\delta k^2=\frac{2k^2}{I_2}\int_{R_1}^{R_2}\Big[&-\left\{\frac{i}{k}\,(\sigma^2-\Omega_A{}^2)\frac{\mathrm{d}}{\mathrm{d}r}\,(r\xi_r)+\frac{2m\sigma\Omega}{ik}\,\xi_r\right\}\delta\xi_z k^2\\ &+\frac{f(\sigma)}{\sigma^2-\Omega_A{}^2}\,\frac{r}{2}\delta k^2\xi_r{}^2+k^2r(\sigma^2-\Omega_A{}^2)\xi_z\delta\xi_z\\ &+\frac{m^2}{r}\,(\sigma^2-\Omega_A{}^2)\xi_z\delta\xi_z\Big]\,\mathrm{d}r. \end{aligned}$$

So that

$$\delta k^2\left[m^2-\frac{k^2}{I_2}\int_{R_1}^{R_2}\frac{f(\sigma)}{\sigma^2-\Omega_A^2}\, r\xi_r^2 dr\right]$$

$$=\frac{2k^2}{I_2}\int_{R_1}^{R_2}\Big[-ik(\sigma^2-\Omega_A^2)\frac{d}{dr}(r\xi_r)-2m\sigma\Omega\frac{k}{i}\xi_r+k^2r(\sigma^2-\Omega_A^2)\xi_z$$

$$+\frac{m^2}{r}(\sigma^2-\Omega_A^2)\xi_z\Big]\,\delta\xi_z\,dr.$$

Consequently, $\delta k^2=0$ for an arbitrary variation $\delta\xi_z$ compatible with the boundary conditions (10) if

$$ik(\sigma^2-\Omega_A^2)\frac{d}{dr}(r\xi_r)-i2m\sigma\Omega k\xi_r=\left(k^2r+\frac{m^2}{r}\right)(\sigma^2-\Omega_A^2)\xi_z. \tag{17}$$

Conversely, whenever equation (17) is satisfied $\delta k^2=0$. But equation (17) is the same as equation (7). Therefore, solving the characteristic value problem presented by (7), (8) and (9) is equivalent to finding a maximum or minimum of

$$I_1=\int_{R_1}^{R_2}\left[\frac{-f(\sigma)}{\sigma^2-\Omega_A^2}\,\xi_r^2+(\sigma^2-\Omega_A^2)\xi_z^2\right] r dr, \tag{18}$$

for given

$$I_2=\int^{R_2}(\sigma^2-\Omega_A^2)\xi_z^2\,\frac{dr}{r}, \tag{19}$$

for arbitrary variation $\delta\xi_z$ subject only to the boundary conditions (10) and the ratio I_1/I_2 at such a maximum or a minimum is a characteristic of k^2.

From (5) it is clear that for real $\sigma(=p+m\Omega)$ such that $f(\sigma)\geqslant 0$, k^2 cannot admit a positive characteristic value. Also, from (9), for real values of σ such that $f(\sigma)<0$, k^2 cannot admit a positive characteristic value. Therefore, for all real values of σ, k^2 cannot admit a positive characteristic value. For real p's the characteristic values of k are necessarily imaginary Therefore, for real k's, p must necessarily be complex; and this means instability.

In case $\Omega_A=0$, *i.e.*, in the corresponding nonmagnetic problem considered by Chandrasekhar (1960b),

$$f(\sigma)=\sigma^2(\sigma^2-\phi). \qquad \ldots \tag{20}$$

So that $f(\sigma)>0$ for real values of σ, if $\phi<0$ Hence, as discussed in the above, (5) leads to a criterion of instability. But for positive values of ϕ such that $\sigma^2<\phi$, $f(\sigma)<0$ and by (9) the system is again unstable. This contradicts the conclusion derived by Chandrasekhar (1960b) that for $\sigma^2<\phi$, (ϕ being everywhere positive), system is stable. The reason for the same is obvious. For, if $\phi>0$ such that

$\sigma^2 < \phi, f(\sigma) < 0$ but the numerator on the right of (5) is not necessarily negative due to the presence of the factor involving $\tilde{\omega}$. The characteristic values of k should therefore be determined from (9) whenever $f(\sigma) < 0$. The above results are of very general nature and to have a precise criterion for stability we have to proceed otherwise.

3. Solution of the Perturbation Equations

Eliminating $\tilde{\omega}$ between (1) and (2), we get

$$\frac{d^2\xi_r}{dr^2} + P\frac{\mathrm{d}}{\mathrm{d}r}\xi_r + Q\xi_r = 0, \qquad \dots \quad (21)$$

where

$$P = \frac{1}{r} + \frac{2m\sigma}{\sigma^2-\Omega_A{}^2}\frac{\mathrm{d}}{\mathrm{d}r}\Omega + \frac{2m^2}{r(m^2+k^2r^2)}$$

and

$$Q = -\frac{1}{r^2} + \frac{2m^2}{r^2(m^2+k^2r^2)} + \frac{2m\Omega}{r(\sigma^2-\Omega_A{}^2)}\left(\sigma/r - m\frac{\mathrm{d}}{\mathrm{d}r}\Omega\right) + \frac{2m\sigma\Omega(-m^2+r^2k^2)}{(\sigma^2-\Omega_A{}^2)r^2(m^2+k^2r^2)}$$

$$-\frac{4m^2\sigma^2\Omega^2}{r^2(\sigma^2-\Omega_A{}^2)^2} - \left(\sigma^2-\Omega_A{}^2-\phi-\frac{4\Omega^2\Omega_A{}^2}{(\sigma^2-\Omega_A{}^2)}\right)\frac{(m^2+k^2r^2)}{r^2(\sigma^2-\Omega_A{}^2)}.$$

Now, applying the transformation

$$\xi_r = \eta_r \exp\left[\tfrac{1}{2}\int(1-Pr)\frac{\mathrm{d}r}{r}\right] \qquad (22)$$

the equation (21) transforms into

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{1}{r^2}\right)\eta_r = g(r)\eta_r, \qquad (23)$$

where

$$g(r) = \tfrac{1}{2}\frac{\mathrm{d}P}{\mathrm{d}r} - \frac{3}{4r^2} + \tfrac{1}{4}P^2 - Q. \qquad (24)$$

Then multiplying both sides of (23) by

$$rC_1(r\xi_i) = r[J_1(r\xi_i)Y_1(R_2\xi_i) - J_1(R_2\xi_i)Y_1(r\xi_i)] \qquad (25)$$

where ξ_i is a root of the transcendental equation

$$J_1(R_1\xi_i)Y_1(R_2\xi_i) - J_1(R_2\xi_i)Y_1(R_1\xi_i) = 0, \qquad (26)$$

and integrating with respect to r from R_1 to R_2 we get

$$\int_{R_1}^{R_2} r\left[\frac{d^2}{dr^2}\eta_r+\frac{1}{r}\frac{d}{dr}\eta_r-\frac{1}{r^2}\eta_r\right] C_1(r\,\xi_i)dr.$$

$$=\int_{R_1}^{R_2} rg(r)C_1(r\xi_i)\eta_r dr. \qquad \dots \quad (27)$$

Following Sneddon (1951, p. 90, eq (74)) this reduces to

$$\int_{R_1}^{R_2}[\xi_i^2+g(r)]rC_1(r\xi_i)\eta_r dr = 0. \qquad \dots \quad (28)$$

As this is true for arbitrary perturbations η_r, we must have

$$\xi_i^2+g(r) = 0. \qquad \dots \quad (29)$$

This is the required characteristic equation.

Stability for symmetric perturbations: When $m = 0$, the equation (21) governing ξ_r reduces to

$$(DD_*-k^2)\xi_r = -\frac{k^2}{K}\left[\phi+4\frac{\Omega^2\Omega_A^2}{K}\right]\xi_r, \qquad \dots \quad (30)$$

where $$K = p^2-\Omega_A^2.$$

This case has been explored by Chandrasekhar (1961, p. 389). He has derived the criterion that in the limit of zero magnetic field, a sufficient condition for stability is that the angular speed, $|\Omega|$, is a monotonic increasing function of r. At the same time, any adverse gradient of angular velocity can be stabilized by a magnetic field of sufficient strength. In the following lines we shall obtain the minimum field strength required to stabilize the system.

In this case the characteristic equation (29) reduces to

$$K^2(\xi_i^2+k^2)-k^2\phi K-4\Omega_A^2\Omega^2k^2 = 0. \qquad \dots \quad (31)$$

So that

$$K = p^2-\Omega_A^2 = \frac{k^2\phi\pm k(k^2\phi^2+16\Omega^2\Omega_A^2(\xi_i^2+k^2))^{\frac{1}{2}}}{2(\xi_i^2+k^2)}.$$

For stability it is necessary that

$$p^2 = \Omega_A^2+\frac{1}{2(\xi_i^2+k^2)}[k^2\phi-k(k^2\phi^2+16\Omega^2\Omega_A^2(\xi_i^2+k^2))^{\frac{1}{2}}] > 0.$$

Therefore, for stability we must have

$$\Omega_A^2 > \frac{k}{2(\xi_i^2+k^2)}[(k^2\phi^2+16\Omega_A^2\Omega^2(\xi_i^2+k^2))^{\frac{1}{2}}-k\phi]. \qquad \dots \quad (32)$$

This is equivalent to

$$\Omega_A^2 > \frac{k^2}{(\xi_i^2+k^2)}(4\Omega^2-\phi). \tag{33}$$

Thus

$$\Delta = \frac{\Omega_A^2{}_{\min}}{4\Omega^2-\phi} = \frac{a^2}{(1-\eta)^2\alpha_i^2} \tag{34}$$

where $k = a/d$, $d = R_2-R_1, \eta = R_1/R_2$ and α_i are the zeros of

$$J_1(\alpha\eta)Y_1(\alpha)-J_1(\alpha)Y_1(\alpha\eta). \tag{35}$$

The behaviours of Δ for the first mode of stability ($i = 1$, $\eta = 0, 0.5$) are shown in figure 1.

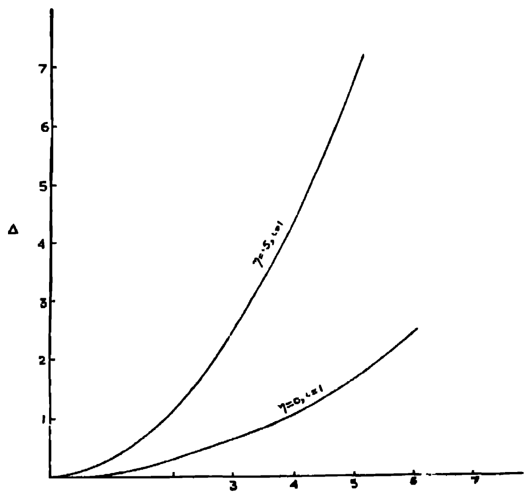

FIG. 1- THE BEHAVIOUR OF $\Delta = \Omega_A^2 \min/(4\Omega^2-\phi)$ FOR THE FIRST MODE OF STABILITY ($i = 1$)

4. Plane Layer Approximation

As pointed out by Edmonds (1958) that when the cylinders are counter-rotating or rotating with greatly differing angular velocities, it is desirable to take some account of the variation of $V(r)$ with r. A better set of approximations are (Edmonds 1958, p 33 equation (39)) :

$$A+B/r^2 \simeq \Omega_1\left[1+\alpha(2-\eta)\left(\frac{r-R_1}{d}\right)\right]$$

$$B/r^2 \simeq \frac{\Omega_1\alpha R_2^2}{R_1^2-R_2^2}\left[1-2\frac{d}{R_1}\left(\frac{r-R_1}{d}\right)\right] \qquad \ldots \quad (36)$$

and $$A \simeq \Omega_1\left[1-\frac{\alpha R_2^2}{R_1^2-R_2^2}\right]$$

where $d = R_2-R_1 \ll \frac{1}{2}(R_2+R_1) = R_0$, $\alpha = \mu-1$, $\mu = \Omega_2/\Omega_1$, $\eta = R_1/R_2$,

instead of approximations used by Chandrasekhar (1961 p. 391 equation (89)). With the above approximations the equations (30) governing ξ_r reduces to

$$(\mathrm{D}^2+\alpha+\beta\zeta+\gamma\zeta^2)\xi_r = 0, \qquad \ldots \quad (37)$$

where $$\mathrm{D} \equiv \frac{\mathrm{d}}{\mathrm{d}\zeta}, \quad \zeta = (r-R_1)/d,$$

$$\alpha = -a^2+4a^2\frac{\Omega_1^2}{K^2}\left\{\frac{\eta^2-\mu}{\eta^2-1}p^2+\Omega_A^2\frac{(\mu-1)}{\eta^2-1}\right\},$$

$$\beta = 4a^2\Omega_1^2\frac{(\mu-1)}{K^2}\left\{\frac{2\Omega_A^2}{\eta(\eta+1)}+\frac{(2-\eta)(\eta^2-\mu)}{\eta^2-1}p^2+\frac{\Omega_A^2(\mu-1)(2-\eta)}{\eta^2-1}\right\} \ldots \quad (38)$$

and $$\gamma = 8a^2\frac{\Omega_1^2\Omega_A^2}{K^2}\frac{(\mu-1)^2}{\eta(\eta+1)}(2-\eta).$$

The equation (37) by a simple change of variable reduces to the well-known confluent hypergeometric equation

$$(\mathrm{D}^2-\tfrac{1}{4}z^2+\nu+\tfrac{1}{2})\xi_r = 0, \qquad \ldots \quad (39)$$

where $z = \sqrt{2}e^{-\frac{1}{8}i\pi}\gamma^{1/4}(\zeta+\beta/2\gamma)$ and $\nu+\frac{1}{2} = \frac{i}{2\gamma^{\frac{1}{2}}}\left(\alpha-\frac{\beta^2}{4\gamma^2}\right)$.

The boundary conditions require

$$\xi_r = 0 \quad \text{at} \quad z = \frac{1}{\sqrt{2}}e^{-\frac{1}{8}i\pi}\beta\gamma^{-3/4} = z_1 \qquad \ldots \quad (40)$$

and $$z = \gamma^{1/4}\sqrt{2}e^{-\frac{1}{8}i\pi}(1+\beta/2\gamma) = z_2. \qquad \ldots \quad (41)$$

The solutions of the equation (34) can be expressed in terms of the parabolic cylinder functions (Erdélyi 1953, p. 117) in the form

$$\xi_r = AD_\nu(z)+BD_{-\nu-1}(iz). \qquad \ldots \quad (42)$$

The boundary conditions, then, require

$$\frac{D_\nu(z_2)}{D_{-\nu-1}(iz_2)} = \frac{D_\nu(z_1)}{D_{-\nu-1}(iz_1)}\,. \tag{43}$$

In the following lines we shall try to establish the above (z_1, z_2)-relationship.

The solutions of (42)

Using the relation (Erdélyi 1953, p. 122)) $D_\nu(z) \sim z^\nu e^{-\frac{1}{4}z^2}$ for large values of $|z|$, $-3/4\pi < \arg z < 3/4\pi$, the equation (43) takes the form

$$\left(\frac{z_2}{z_1}\right)^{2\nu+1} e^{-\frac{1}{4}(z_2^2 - z_1^2)} = 1.$$

So that

$$\frac{z_2}{z_1}\left[1 - \frac{z_2^2 - z_1^2}{2(2\nu+1)}\right] = 1,$$

where terms of the second and higher orders of $\dfrac{z_2^2 - z_1^2}{2(2\nu+1)}$ have been neglected.

Thus the (z_1, z_2)-relationship is determined from the cubic equation

$$z_2^3 - [2(2\nu+1) + z_1^2]z_2 + 2(2\nu+1)z_1 = 0. \tag{44}$$

The solutions of this equation are given by

$$z_1 = z_2 \tag{45}$$

and

$$z_1 z_2 = 4\nu + 2 - z_2^2.$$

Using the transformations

$$z_1 = \sqrt{2\nu+1}\, x \quad \text{and} \quad z_2 = \sqrt{2\nu+1}\, y, \tag{46}$$

the above equations transform into

$$y = x \tag{47}$$

and

$$y^2 + xy - 2 = 0.$$

The former equation represents a straight line passing through the origin and inclined at an angle $\pi/4$ to the x-axis while the latter represents a hyperbola having origin as its centre, the axis being inclined at an angle $3\pi/8$ to the x-axis, the axes being of lengths $2(\sqrt{2}-1)^{\frac{1}{2}}$ and $2(\sqrt{2}+1)^{\frac{1}{2}}$ and the asymptotes are $y = 0$ and $x + y = 0$. The (x, y)-relationship, thus, determined is shown in the figure 2. Now, (40) and (41) give

$$\frac{\beta}{\gamma} = \frac{2z_1}{z_2 - z_1}\,. \qquad \ldots \tag{48}$$

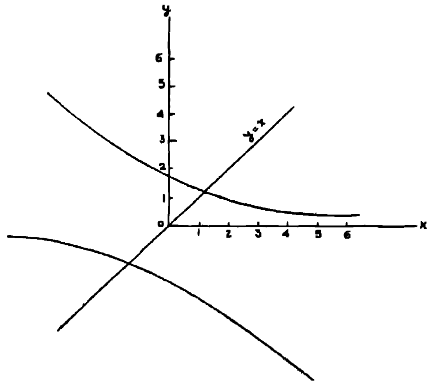

FIG. 2. THE SOLUTIONS OF $D_\nu(z_2)/D_{-\nu-1}(iz_2) = D_\nu(z_1)\, D_{-\nu-1}(iz_1)$

Substituting the value of β/γ from (38) we see that the eigenvalues are given by

$$p^2 = \left[\frac{2z_1}{z_2 - z_1} - \frac{\eta}{2(\eta-1)} - \frac{1}{(\mu-1)(2-\eta)} \right] \frac{2(\mu-1)(\eta-1)}{\eta(\eta^2-\mu)}\, \Omega_A{}^2, \qquad \dots \quad (49)$$

where z_1 and z_2 are determined by (45) through (46) and (47).

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